

The University of Chicago

CONJUGATE OSCULATING QUADRICS
ASSOCIATED WITH THE LINES
OF CURVATURE

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF
PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1936

By

RUTH BEATRICE RASMUSEN

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TABLE OF CONTENTS

Section	Page
1. Introduction	1
2. Analytic Basis	3
3. The Conjugate Osculating Quadrics	10
4. Asymptotic Tangent Planes	17
5. Other Properties	25
6. The Superosculating Lines	28



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I. INTRODUCTION

The two conjugate osculating quadrics at a point of a curve on a surface in ordinary space, relative to a given conjugate net on the surface, were first defined by Lane.¹ The lines of curvature on a surface are a conjugate net, and determine at each point of a curve on the surface two conjugate osculating quadrics which have interesting metric properties. The purpose of this thesis is to investigate these properties.

The definition of the conjugate osculating quadrics at a point of a curve on a conjugate net on a surface may be found in Section 2, in which the analytic basis for the subsequent discussion is summarized. The equations of the conjugate osculating quadrics associated with the lines of curvature at a point of a curve are derived in Section 3. In this section these equations are employed to show that the conjugate osculating quadrics at a point of a curve on the lines of curvature are either hyperboloids of one sheet or hyperbolic paraboloids. A new characterization of surfaces of Monge (surfaces whose lines of curvature in one system are geodesics) is given by means of these quadrics.

Section 4 is devoted to a study of the properties of the conjugate osculating quadrics on the lines of curvature at a point

¹E. P. Lane, Conjugate Nets and the Lines of Curvature, American Journal of Mathematics, vol. LIII (1931), p. 577.

of a curve on a surface which are associated with the asymptotic tangent planes (tangent planes at infinity) of the ruled surfaces each of which is formed by drawing the tangents of the curves of one family of the lines of curvature at each fixed point of the given curve.

There are two pairs of quadrics intimately associated with the conjugate osculating quadrics at a point of a curve on the lines of curvature. A quadric of one pair is introduced in Section 4 and may be defined in the following manner. The center of a conjugate osculating quadric at a point of a curve on the lines of curvature lies on three particular planes. Two of these planes vary as the curve defining them varies but remains tangent to a fixed line. The locus of the line of intersection of these two planes is a quadric surface. The third plane is the asymptotic tangent plane of one of the ruled surfaces associated with the curve which defines the quadric. A quadric of the second pair is defined in Section 5 as the locus of the conic of intersection of one of the conjugate osculating quadrics at a point of a curve and the osculating plane of the curve at the point as the curve varies but remains tangent to a fixed line.

It is shown in Section 6 that when a surface is referred to its lines of curvature the directions of the superosculating lines on the surface are defined by equating to zero the terms of third degree in the series representing the surface. The superosculating lines and the curves of Darboux on a surface coincide if and only if the surface has constant total curvature.

2. ANALYTIC BASIS

This section begins with the definition of a conjugate osculating quadric at a point of a curve on a given conjugate net on a surface followed by the equation of the conjugate osculating quadric in local projective homogeneous coordinates. Then an analytic basis is established for the metric differential geometry of the lines of curvature taken as a parametric net on a surface in ordinary space. Some portions of this geometry, which will be needed later on, are summarized. The summary also includes a transformation which has been devised, from the local projective homogeneous coordinates which are used in the projective theory of a general conjugate net, to the local Cartesian non-homogeneous coordinates which are used in the metric theory of the lines of curvature. This transformation is applied to the equation of the conjugate osculating quadric in local projective homogeneous coordinates to obtain the equation in local Cartesian non-homogeneous coordinates. The equation thus obtained, having its coefficients expressed partially in terms of the invariants of the metric theory, is more convenient than the equation obtained in Section 3 which has its coefficients expressed partially in terms of the Christoffel symbols.

A curve C on a surface S at a point P is defined to be the curve whose direction is given by

$$dv - \lambda du = 0.$$

If on a conjugate net N on a non-developable surface in ordinary projective space, C is any curve through the point P distinct

from the curves of the net, the conjugate osculating quadric at the point P of the curve C on the net N on the surface S is defined to be the limit of the quadric determined by three tangents of one family of the net N constructed at the point P and at two neighboring points P_1, P_2 of C as P_1, P_2 independently approach the point P along C. In a similar manner, three tangents of the other family of the net N determine a conjugate osculating quadric at the point P of C on N. The symbols Q_u and Q_v denote the conjugate osculating quadrics determined by the u-tangents and the v-tangents, respectively. The equation of the conjugate osculating quadric¹ Q_u , referred to the tetrahedron x, ρ, σ, y is

$$(1) \quad (L - N\lambda^2)x_3^2 - \lambda[4(\mathcal{C}' + \lambda\mathcal{B}') - (\log \lambda r^{\frac{1}{2}})']x_3x_4 \\ + \lambda(2n\lambda - \lambda^2K/L + H/N)x_4^2 + 2\lambda(\lambda x_1x_4 - Lx_2x_3) = 0,$$

where

$$\lambda' = \lambda_u + \lambda\lambda_v.$$

In the metric differential geometry of a non-developable surface which is not a sphere in ordinary space, the lines of curvature are the only orthogonal conjugate net. For this reason, it is convenient to take the lines of curvature for parametric curves and to employ a local trihedron of reference at a point of the surface, whose edges are the tangents of the lines of curvature, and the normal at the point of the surface.

Let us consider in ordinary metric space a surface whose parametric equations in Cartesian coordinates are

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

where x, y, z are single-valued analytic functions of two inde-

¹Ibid., p. 579.

pendent variables u, v . Suppose that the lines of curvature have been taken for parametric curves. Then the first and second fundamental forms are

$$E du^2 + G dv^2, \quad D du^2 + D'' dv^2.$$

The coordinates x, y, z of a variable point of a surface and the direction cosines X, Y, Z of its surface normal satisfy the equations of Gauss

$$(2) \quad \begin{aligned} x_{uu} &= \{11, 1\} x_u + \{11, 2\} x_v + DX, \\ x_{uv} &= \{12, 1\} x_u + \{12, 2\} x_v, \\ x_{vv} &= \{22, 1\} x_u + \{22, 2\} x_v + D''X, \end{aligned}$$

and the equations of Rodrigues

$$(3) \quad X_u = -x_u/R_1, \quad X_v = -x_v/R_2.$$

The Christoffel symbols have the following values:

$$(4) \quad \begin{aligned} 2\{11, 1\} &= (\log E)_u, & 2\{22, 2\} &= (\log G)_v, \\ 2\{12, 1\} &= (\log E)_v, & 2\{12, 2\} &= (\log G)_u, \\ 2\{22, 1\} &= -G_u/E, & 2\{11, 2\} &= -E_v/G. \end{aligned}$$

The principal radii of normal curvature R_1, R_2 at a point of the surface are defined by the formulas

$$(5) \quad R_1 = E/D, \quad R_2 = G/D''.$$

The corresponding radii of geodesic curvature of the lines of curvature, C_u and C_v , at the point are given by

$$(6) \quad 1/\rho_1 = -(\log E)_v/2G^{\frac{1}{2}}, \quad 1/\rho_2 = (\log G)_u/2E^{\frac{1}{2}}.$$

The integrability conditions of equations (2) and (3) are the equation of Gauss

$$(7) \quad -2(EG)^{\frac{1}{2}}K_t = [G_u/(EG)^{\frac{1}{2}}]_u + [E_v/(EG)^{\frac{1}{2}}]_v,$$

where the total curvature K_t is defined by

$$(8) \quad K_t = 1/R_1R_2,$$

and the two equations of Codazzi

$$(9) \quad D_v = -DG^{\frac{1}{2}}/\rho_1 - D''E/\rho_1 G^{\frac{1}{2}}, \quad D''_u = DG/\rho_2 E^{\frac{1}{2}} + D''E^{\frac{1}{2}}/\rho_2.$$

Henceforth we shall use at a point P of a surface a local trihedron of reference which has the given point P as origin, the ξ -axis along the u -tangent, the η -axis along the v -tangent of the lines of curvature through the point P and the ζ -axis along the surface normal at P . If $\bar{x}, \bar{y}, \bar{z}$ are the general coordinates of a point having local coordinates ξ, η, ζ , the equations of transformation between the general and the local Cartesian systems can be written in the form

$$(10) \quad \begin{aligned} \bar{x} - x &= \xi x_u/E^{\frac{1}{2}} + \eta x_v/G^{\frac{1}{2}} + \zeta X, \\ \bar{y} - y &= \xi y_u/E^{\frac{1}{2}} + \eta y_v/G^{\frac{1}{2}} + \zeta Y, \\ \bar{z} - z &= \xi z_u/E^{\frac{1}{2}} + \eta z_v/G^{\frac{1}{2}} + \zeta Z. \end{aligned}$$

The equations of transformation which connect the local homogeneous coordinates of the projective theory which are based on the tetrahedron whose vertices are x, ρ, σ, y with suitably chosen unit point, with the local Cartesian non-homogeneous coordinates of the metric theory described in the preceding paragraph are

$$(11) \quad \begin{aligned} x_1 &= 1 + \xi/\rho_2 - \eta/\rho_1 + (\zeta/2) [(R_1 + R_2 + R_2\rho_{1v}/G^{\frac{1}{2}})/\rho_1^2 \\ &\quad + (R_2 + R_1 - R_1\rho_{2u}/E^{\frac{1}{2}})/\rho_2^2], \\ x_2 &= (\xi + R_2\zeta/\rho_2)/E^{\frac{1}{2}}, \quad x_3 = (\eta - R_1\zeta/\rho_1)/G^{\frac{1}{2}}, \quad x_4 = \zeta. \end{aligned}$$

For the lines of curvature the invariants appearing in the projective theory become

$$\begin{aligned}
 H &= G^{\frac{1}{2}}(1/\rho_1)_u, & K &= -E^{\frac{1}{2}}(1/\rho_2)_v, \\
 H &= H + (\log R_1)_{uv}, & K &= K + (\log R_2)_{uv}, \\
 8\mathcal{B}' &= (\log R_1^3/R_2)_v, & 8\mathcal{C}' &= (\log R_2^3/R_1)_u, \\
 \mathcal{D} &= D[(R_2 - R_1 + R_2\rho_{1v}/G^{\frac{1}{2}})/\rho_1^2 \\
 &\quad + (R_2 - R_1 + R_1\rho_{2u}/E^{\frac{1}{2}})/\rho_2^2], \\
 r &= D''/D.
 \end{aligned}
 \tag{12}$$

The equations of the osculating planes of the lines of curvature C_u , C_v at a point of a surface are respectively

$$(13) \quad \rho_1\eta - R_1\xi = 0, \quad \rho_2\xi + R_2\zeta = 0,$$

where ρ_1 , ρ_2 are given by (6), and R_1 , R_2 are given by (5).

The equation of the osculating plane of the curve C is

$$\begin{aligned}
 (14) \quad & (D + D''\lambda^2)(\lambda G^{\frac{1}{2}}\xi - E^{\frac{1}{2}}\eta) + \{(D + D''\lambda^2)(\lambda G^{\frac{1}{2}}R_2/\rho_2 + E^{\frac{1}{2}}R_1/\rho_1) \\
 & + \lambda(EG)^{\frac{1}{2}}[4(\mathcal{C}' - \lambda\mathcal{B}') + \{\log(\lambda D''^{\frac{1}{2}}/D^{\frac{1}{2}})\}']\}\zeta = 0,
 \end{aligned}$$

where the invariants $\mathcal{B}'$, $\mathcal{C}'$ are defined by (12).

In the projective theory the local equations of the tangent¹ to a curve C at a point P of a surface S are

$$(15) \quad x_3 - \lambda x_2 = x_4 = 0.$$

In the metric theory the local equations of the tangent² are

$$(16) \quad \zeta = 0, \quad \eta = \xi \tan \theta,$$

where θ is the angle which the tangent makes with the positive

¹Ibid., p. 580.

²E. P. Lane, Projective Differential Geometry of Curves and Surfaces, The University of Chicago Press, 1932, p. 238.

arm of the ξ -axis. Applying the transformation given by (11) to equation (15), we obtain

$$\zeta = \eta - \lambda(G/E)^{\frac{1}{2}}\xi = 0.$$

Hence we have

$$(17) \quad \lambda = (E/G)^{\frac{1}{2}} \tan \theta.$$

Hereafter whenever λ appears in an equation, it is understood to have the value given by (17). Applying the transformation given by (11) to equation (1) and recalling that the invariants appearing in the coefficients of equation (1) assume the values given in (12) under transformation (11), we have the result:

The local equation of the conjugate osculating quadric Q_u at a point P of a curve C on the lines of curvature is

$$(18) \quad \psi \eta^2/G + \phi \zeta^2 + 2\alpha \eta \zeta + 2\lambda \gamma \xi \zeta - 2\lambda DT \xi \eta + 2\lambda^2 \zeta = 0,$$

where

$$(19) \quad \begin{aligned} \psi &= D - \lambda^2 D'', \quad T = 1/(EG)^{\frac{1}{2}}, \quad \gamma = \lambda/\rho_2 + E^{\frac{1}{2}}/\rho_1 G^{\frac{1}{2}}, \\ 2\alpha &= -2R_1\psi/\rho_1 G - \lambda[4(\mathfrak{C}') + \lambda \mathfrak{B}'] - \{\log(\lambda D''^{\frac{1}{2}}/D^{\frac{1}{2}})\}' / G^{\frac{1}{2}} \\ &\quad - 2\lambda^2/\rho_1 - 2\lambda DR_2 T/\rho_2, \\ \phi &= -2\alpha R_1/\rho_1 - R_1^2 \psi/\rho_1^2 G - \lambda^3 K/D + \lambda H/D'' \\ &\quad + 2\lambda^2(R_1 - R_1 \rho_{2u}/E^{\frac{1}{2}})/\rho_2^2. \end{aligned}$$

If the substitution

$$\begin{pmatrix} u & \xi & E & D & \lambda & \lambda' \\ v & \eta & G & D'' & 1/\lambda & -\lambda'/\lambda^3 \end{pmatrix}$$

is made in equation (18), the local equation of the conjugate osculating quadric Q_v at a point P of a curve C on the lines of curvature becomes

$$(20) \quad \begin{aligned} \psi \xi^2/E + \pi \zeta^2 + 2\lambda D''T \xi \eta + 2\beta \xi \zeta \\ + 2\gamma G^{\frac{1}{2}} \eta \zeta / E^{\frac{1}{2}} - 2\zeta = 0, \end{aligned}$$

where

$$\begin{aligned} 2\beta = 2R_2\psi/\rho_2E + [4(\mathcal{C}' + \lambda\mathcal{B}') + \{\log(\lambda D''^{\frac{1}{2}}/D^{\frac{1}{2}})\}'] / E^{\frac{1}{2}} \\ - 2/\rho_2 - 2\lambda D''R_1T/\rho_1, \end{aligned}$$

$$\begin{aligned} \pi = 2R_2\beta/\rho_2 + H/\lambda D'' - \lambda K/D - (2R_2 + 2R_2\rho_{1v}/G^{\frac{1}{2}})/\rho_1^2 \\ - R_2^2\psi/\rho_2^2E. \end{aligned}$$

3. THE CONJUGATE OSCULATING QUADRICS

In this section the local equations in Cartesian non-homogeneous coordinates of the conjugate osculating quadrics Q_u , Q_v at a point P of a curve C on the lines of curvature taken as parametric curves on a surface are obtained. These equations are shown to be identical with the equations which were obtained at the end of Section 2. By means of equations (18) and (20) the conjugate osculating quadrics at a point of a curve on the lines of curvature are classified and a characterization of surfaces of Monge is obtained.

The coordinate system is restricted to be real since the lines of curvature are real on a real surface.

The definition of the conjugate osculating quadrics associated with the lines of curvature can be obtained from the projective definition given in Section 2 by replacing the net N by the lines of curvature. In the metric case the integral surface is not a sphere, as well as non-developable.

We now set out to find the equation of the conjugate osculating quadric Q_u determined by three consecutive u -tangents of the lines of curvature at a point P of a curve C of the family represented by the equation

$$dv - \lambda du = 0$$

on the lines of curvature which are taken as parametric curves on an integral surface S of system (2), (3). From system (2), (3) by differentiation and substitution one obtains

$$\begin{aligned}
 x_{uuu} &= [\{11, 1\}_u + \{11, 1\}^2 + \{11, 2\}\{12, 1\} - D/R_1]x_u \\
 &\quad + [\{11, 2\}_u + \{11, 1\}\{11, 2\} + \{11, 2\}\{12, 2\}]x_v \\
 &\quad + (D_u + D\{11, 1\})x, \\
 x_{uuv} &= [\{12, 1\}_u + \{12, 1\}\{11, 1\} + \{12, 2\}\{12, 1\}]x_u \\
 (21) \quad &\quad + [\{12, 2\}_u + \{12, 1\}\{11, 2\} + \{12, 2\}^2]x_v \\
 &\quad + D\{12, 1\}x, \\
 x_{uvv} &= [\{12, 1\}_v + \{12, 1\}^2 + \{12, 2\}\{22, 1\}]x_u \\
 &\quad + [\{12, 2\}_v + \{12, 1\}\{12, 2\} + \{12, 2\}\{22, 2\}]x_v \\
 &\quad + D''\{12, 2\}x.
 \end{aligned}$$

Similarly, any derivative of x can be expressed as a linear combination of x_u , x_v , x . Any point Y on the curve C and near the point P can be defined by a power series in the increment Δu , corresponding to displacement on C from P to the point Y , of the form

$$Y = x + x' \Delta u + x'' \Delta u^2 / 2 + \dots,$$

where the accents indicate total derivatives with respect to u .

Then we find

$$Y = x + \xi x_u / E^{\frac{1}{2}} + \eta x_v / G^{\frac{1}{2}} + \zeta x,$$

where

$$\begin{aligned}
 \xi &= (\Delta u + \dots) E^{\frac{1}{2}} \\
 (22) \quad \eta &= (\lambda \Delta u + [\{11, 2\} + 2\lambda\{12, 2\} + \lambda^2\{22, 2\} + \lambda'\Delta u^2/2 + \dots] G^{\frac{1}{2}}), \\
 \zeta &= (D + \lambda^2 D'') \Delta u^2 / 2 + \dots
 \end{aligned}$$

These series represent the local coordinates ξ , η , ζ of the point Y , referred to the local trihedron whose edges are the tangents of the lines of curvature and the surface normal, to terms of as high degree as will be needed. Similarly, expanding Y_u , we have

$$\begin{aligned}
 Y_u &= x_u + (x_u)' \Delta u + (x_u)'' \Delta u^2/2 + \dots \\
 &= x_u + (x_{uu} + x_{uv} \lambda) \Delta u + (x_{uuu} + 2x_{uuv} \lambda + x_{uvv} \lambda^2 \\
 (23) \quad &\quad + x_{uv} \lambda') \Delta u^2/2 + \dots \\
 &= \xi x_u/E^{\frac{1}{2}} + \eta x_v/G^{\frac{1}{2}} + \zeta X,
 \end{aligned}$$

where

$$\begin{aligned}
 \xi &= E^{\frac{1}{2}} [1 + (\{11, 1\} + \lambda \{12, 1\}) \Delta u + \dots], \\
 \eta &= G^{\frac{1}{2}} \{ (\{11, 2\} + \lambda \{12, 2\}) \Delta u + [\{11, 2\}_u + \{11, 1\} \{11, 2\} \\
 &\quad + \{11, 2\} \{12, 2\} + 2\lambda (\{12, 2\}_u + \{12, 1\} \{11, 2\} + \{12, 2\}^2) \\
 (24) \quad &\quad + \lambda^2 (\{12, 2\}_v + \{12, 2\} \{12, 1\} + \{12, 2\} \{22, 2\}) \\
 &\quad + \lambda' \{12, 2\}] \Delta u^2/2 + \dots \}, \\
 \zeta &= D \Delta u + [D_u + D \{11, 1\} + 2\lambda D \{12, 1\} + \lambda^2 D'' \{12, 2\}] \Delta u^2/2 \\
 &\quad + \dots .
 \end{aligned}$$

The vector equation of the u-tangent at a point of the curve C is

$$Z = Y + Y_u t/E^{\frac{1}{2}},$$

where Y represents the coordinates of the point of contact; t is the distance of the point of contact from a variable point Z on the tangent and $Y_u/E^{\frac{1}{2}}$ are the direction cosines of the u-tangent.

In order to calculate the power series expansions for the local coordinates ξ, η, ζ of any point $Y + tY_u/E^{\frac{1}{2}}$ on the u-tangent at the point Y, it is sufficient to multiply the series (22) by one and the series (24) by $t/E^{\frac{1}{2}}$ and add corresponding terms.

Then writing the equation of the most general quadric surface and demanding that it be satisfied by the power series thus calculated, identically in t and in Δu as far as terms of the second degree, we obtain the equation of the quadric Q_u , namely,

$$\begin{aligned}
 (25) \quad &\eta^2 (D - \lambda^2 D'')/G + a \zeta^2/D + b \eta \zeta/DG^{\frac{1}{2}} + 2\lambda \xi \zeta (\{11, 2\} \\
 &\quad + \lambda \{12, 2\})/E^{\frac{1}{2}} - 2D\lambda \xi \eta/E^{\frac{1}{2}}G^{\frac{1}{2}} + 2\lambda^2 \zeta = 0,
 \end{aligned}$$

where

$$\begin{aligned}
 a &= \lambda^3(\{12, 2\}_v - \{12, 1\}\{12, 2\}) + \lambda^2(2\{12, 2\}_u + 2\{12, 2\}^2 \\
 &\quad - 2\{11, 1\}\{12, 2\} - \{11, 2\}\{22, 2\}) + \lambda(\{11, 2\}_u \\
 &\quad + 2\{11, 2\}\{12, 2\} - \{11, 1\}\{11, 2\}) + \{11, 2\}^2 - \lambda'\{11, 2\}, \\
 b &= \lambda^2(D''\{11, 2\} + D\{22, 2\}) - \lambda(D_u - D\{11, 1\} + D\{12, 2\}) \\
 &\quad - 2D\{11, 2\} + D\lambda'.
 \end{aligned}$$

The equation of the conjugate osculating quadric Q_v at a point P of the curve C on the lines of curvature can be written by interchanging u and v , and making suitable changes in the appropriate symbols. The result is

$$\begin{aligned}
 (26) \quad &\xi^2(D - \lambda^2 D'')/E - c \zeta^2/D - d \xi \zeta / D'' E^{\frac{1}{2}} - 2\eta \zeta (\{12, 1\} \\
 &\quad + \lambda\{22, 1\})/G^{\frac{1}{2}} + 2D'' \lambda \xi \eta / E^{\frac{1}{2}} G^{\frac{1}{2}} - 2\zeta = 0,
 \end{aligned}$$

where

$$\begin{aligned}
 c &= (\{12, 1\}_u - \{12, 2\}\{12, 1\})/\lambda + 2\{12, 1\}_v + 2\{12, 1\}^2 \\
 &\quad - 2\{22, 2\}\{12, 1\} - \{22, 1\}\{11, 1\} + \lambda(\{12, 2\}_v \\
 &\quad + 2\{22, 1\}\{12, 1\} - \{22, 2\}\{22, 1\}) + \lambda^2\{22, 1\}^2 \\
 &\quad + \lambda'\{22, 1\}/\lambda, \\
 d &= D\{22, 1\} + D''\{11, 1\} - \lambda(D''_v - D''\{22, 2\} + D''\{22, 1\}) \\
 &\quad - 2D''\lambda^2\{22, 1\} - D''\lambda'/\lambda.
 \end{aligned}$$

By substituting the values of the Christoffel symbols given in (4) in equations (25) and (26), recalling the values of the invariants for the lines of curvature given in (12) and using the equations of Codazzi given in (9), it can be verified that equations (18) and (25) are identical, as well as equations (20) and (26). It is more convenient to use equations (18), (20) of the quadrics Q_u , Q_v , respectively.

The tangent plane $\zeta = 0$ at the point P of the surface S intersects the quadric Q_u whose equation is given by (18) in the u -tangent $\eta = \zeta = 0$, and in the residual line

$$(27) \quad \psi\eta/G - 2\lambda_{DT}\xi = \zeta = 0.$$

Similarly, the tangent plane intersects the quadric Q_v whose equation is given by (20) in the v -tangent $\xi = \zeta = 0$, and in the residual line

$$(28) \quad \psi\xi/E + 2\lambda_{D''T}\eta = \zeta = 0.$$

Since $\lambda \neq 0, \infty, DD'' \neq 0, EG \neq 0$, it is evident from equations (19) that the residual lines (27) and (28) cannot coincide with the u -tangent and the v -tangent, respectively. Consequently, the conjugate osculating quadrics associated with a point of a curve on the lines of curvature on a surface are either hyperboloids of one sheet or hyperbolic paraboloids.

The invariants D of the quadrics Q_u and Q_v have the values

$$(29) \quad \begin{aligned} d_u &= -2\lambda^3 \alpha_{DT}/\rho_s - \psi(\lambda^4/\rho_s^2 + 2\lambda^3 D_{R_1 T}/\rho_1 \rho_s)/G \\ &\quad - \lambda^2 D^2 T^2 [\lambda H/D'' - \lambda^3 K/D + 2\lambda^2 (R_1 - R_1 \rho_{su}/E^{\frac{1}{2}})/\rho_s^2], \\ d_v &= 2\beta \lambda D'' T/\rho_1 - \psi(1/\rho_1^2 + 2\lambda G^{\frac{1}{2}}/\rho_1 \rho_s E^{\frac{1}{2}})/E \\ &\quad - \lambda^2 D''^2 T^2 [H/\lambda D - \lambda K/D - (2R_s + 2R_z \rho_{1v}/G^{\frac{1}{2}})/\rho_1^2], \end{aligned}$$

where the values of α, T, ψ, β are given by (19) and (20).

Hence, the differential equation of the curves on a surface for which the associated conjugate osculating quadrics Q_u are paraboloids is given by

$$(30) \quad d_u = 0.$$

At point P of a surface S where $1/\rho_s \neq 0$ it is obvious that equation (30) defines a curve in each direction. Therefore, at each point of a surface there is associated with each direction (except $0, \infty$) a one-parameter family of quadrics Q_u . At points where $1/\rho_s \neq 0$ each family consists of hyperboloids of one sheet and just one hyperbolic paraboloid.

If $1/\rho_s = 0$ at a point of a surface equation (30) is independent of λ' . In this case the following theorems are easily deduced:

If $1/\rho_s = 0$, $K \neq 0$, $H \neq 0$ at a point of a surface, there are two directions at such points for which the two associated one-parameter families of conjugate osculating quadrics Q_u on the lines of curvature consist of just hyperbolic paraboloids.

If $1/\rho_s = 0$, $K \neq 0$, $H = 0$ or if $1/\rho_s = 0$, $K = 0$, $H \neq 0$ at a point of a surface on which the lines of curvature are taken as parametric curves, the conjugate osculating quadrics Q_u associated with all curves at such a point are hyperboloids of one sheet.

If $1/\rho_s = 0$, $K = 0$, $H = 0$ at a point of a surface on which the lines of curvature are taken as parametric curves, then the conjugate osculating quadrics Q_u associated with all curves at such a point are hyperbolic paraboloids.

A surface of Monge is a surface whose lines of curvature in one system are geodesics.¹ When the parametric lines upon a surface form an orthogonal system, a necessary and sufficient condition that the curves C_u , or C_v be geodesics is that E be a function of u alone, or G of v alone, respectively.² All curves in the systems C_u and C_v cannot simultaneously be geodesics, for, by the Gauss integrability condition given in (7), $K_t = 0$ at all points of such a surface. If the curves C_u are geodesics, using formulas (12), we find $H = 0$, $\mathbf{H} = 0$; if the curves C_v are geodesics, then $K = 0$, $\mathbf{K} = 0$. Consequently, on a surface of Monge on which the lines of curvature are parametric either $H = 0$,

¹L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Ginn and Company, 1909, p. 306.

²Ibid., p. 134.

$H = 0$, or $K = 0$, $K = 0$ at all ordinary points.

On a surface of Monge on which the curves C_v are geodesics a necessary and sufficient condition that the quadrics Q_u associated with all curves C at P be paraboloids is $H = 0$. From this fact and the theorems stated above we conclude that a surface of Monge is the only surface such that at each point of the surface the conjugate osculating quadrics of one family, say Q_u , are all either hyperboloids of one sheet or hyperbolic paraboloids, while the quadrics of the family Q_v defined at each point for the same curves are both hyperboloids and paraboloids.

If there is a surface on which the v -curves are geodesics, and if $H \neq 0$ everywhere on such a surface, the differential equation of the curves for which the hyperboloids Q_u are surfaces of revolution can be obtained by setting the discriminant of the discriminating cubic of Q_u equal to zero.

Similar results may be obtained for the quadrics Q_v .

4. ASYMPTOTIC TANGENT PLANES

Demoulin¹ investigated the properties of the quadric of Lie at a point of a surface by associating with the quadric at each point of the integral surface the asymptotic tangent planes (tangent planes at infinity) of the two ruled surfaces of asymptotic tangents circumscribing the asymptotics on the surface through the point. The conjugate osculating quadrics have properties which may be studied by associating with each of the conjugate osculating quadrics at a point of a curve on a surface an asymptotic tangent plane of a ruled surface formed by constructing the tangents of the curves of one family of the lines of curvature at the points of the curve which defines the quadrics. Since the purpose of this section is to study these properties, we first find the equations of the asymptotic tangent planes of the ruled surfaces. As stated in the introduction the equation of one of a pair of quadrics defined for each direction at a point of a surface is derived.

We are now going to find the equation of the asymptotic tangent plane of the ruled surface R_u of u -tangents of the lines of curvature constructed at the points of a fixed curve C (distinct from the lines of curvature) on an integral surface of system (2), (3) on which the lines of curvature are parametric. The parametric vector equation of R_u is

$$Z = Y + tY_u/E^{\frac{1}{2}},$$

¹A. Demoulin, Sur la quadrique de Lie, Comptes Rendus, vol. 147 (1908), p. 493.

where the parameters are t and u , and where Z , Y , t have the same meanings as in Section 3. The equation

$$(\xi - Z, Z_t, Z_u) = 0$$

at a point Z of the surface R_u reduces to

$$\lambda(\xi - Y, Y_u, Y_v) + t(\xi - Y, Y_u, Y_{uu} + \lambda Y_{uv}) = 0.$$

It follows that the equation of the tangent plane at infinity is

$$(\xi - Y, Y_u, Y_{uu} + \lambda Y_{uv}) = 0.$$

The local equation of the asymptotic tangent plane of R_u which passes through the point P is obtained by differentiating the series (23) with respect to u , and with respect to v , permitting Δu to approach zero, substituting for x_{uu} , x_{uv} from (2), and finally substituting for the points Y , Y_u , Y_{uu} , Y_{uv} which are now linear combinations of x_u , x_v , X their local coordinates. Upon performing the necessary calculations, the local equation of the asymptotic tangent plane of R_u through P becomes

$$(31) \quad DT\eta - \gamma\zeta = 0,$$

where T and γ are given by (19). By a repetition of the above process, or by making appropriate substitutions in equation (31), the local equation of the asymptotic tangent plane of the ruled surface R_v can be written. It is

$$(32) \quad \lambda\xi/R_s + \gamma\zeta = 0.$$

Recalling that the quadrics Q_u associated with all curves at a point of a surface have real rulings, and, in particular, have the u -tangent at the point as a common generator, one obtains the result:

The quadrics Q_u associated with all curves at a point of a surface have in common an axial pencil of tangent planes. The pencil consists of the tangent plane of the surface at the point, the osculating plane of C_u and the asymptotic tangent planes through the point of the ruled surfaces R_u defined for the curves at the point which are not tangent to the lines of curvature at the point.

This theorem has as a special case a portion of a theorem of Lane.¹

The equation of an asymptotic tangent plane through a generator consecutive to the u -tangent is obtained by setting equal to zero the derivative with respect to u of equation (31). As a first step in obtaining this equation, partial derivatives with respect to u and with respect to v of the local coordinates are convenient. These derivatives are easily obtained by differentiating equation (10₁) with respect to u alone, and with respect to v alone, simplifying by equations (2) and (3), and then demanding that the two equations thus obtained be identities in x_u, x_v, X . The desired formulas are

$$\begin{aligned}
 \xi_u &= E^{\frac{1}{2}}(\eta/\rho_1 + \zeta/R_1 - 1), \\
 \eta_u &= -E^{\frac{1}{2}}\xi/\rho_1, & \zeta_u &= -D\xi/E^{\frac{1}{2}}; \\
 \xi_v &= \eta G^{\frac{1}{2}}/\rho_2, & \eta_v &= G^{\frac{1}{2}}(-\xi/\rho_2 + \zeta/R_2 - 1), \\
 \zeta_v &= -D''\eta/G^{\frac{1}{2}}
 \end{aligned}
 \tag{33}$$

The equation of the asymptotic tangent plane of a ruled surface R_u through a consecutive generator is

$$(34) \quad \eta[\lambda D''\gamma/G^{\frac{1}{2}} + (D\gamma)'] - \zeta(\gamma' - D\lambda/R_2 E^{\frac{1}{2}}) - D\lambda/E^{\frac{1}{2}} = 0,$$

¹E. P. Lane, Conjugate Nets and the Lines of Curvature, American Journal of Mathematics, vol. LIII (1931), p. 580.

where the accents denote derivatives with respect to u except in the case of D'' . The planes (34) form a pencil with axis in the tangent plane for a fixed λ .

The center of the conjugate osculating quadric Q_u on the lines of curvature at a point P of a curve C on a surface S lies on the planes

$$(35) \quad DT\eta - \gamma\zeta = 0,$$

$$(36) \quad \lambda DT\xi - \psi\eta/G - \alpha\zeta = 0,$$

$$(37) \quad \lambda\gamma\xi + \alpha\eta + \phi\zeta + \lambda^2 = 0.$$

The coordinates, when finite, of the center of Q_u are

$$(38) \quad \lambda^3[\psi\gamma/G + DT\alpha]/d_u, \quad \lambda^4\gamma DT/d_u, \quad \lambda^4D^2T^2/d_u,$$

where d_u has the value given in (29). Since equation (35) is independent of λ' , the centers of the quadrics Q_u associated with all curves tangent to a given direction at a point of a surface lie in a plane which is the asymptotic tangent plane through the point of all ruled surfaces R_u associated with the direction.

From the value of γ in equation (19) and from equation (31) we conclude:

On a surface S at points P where $1/\rho_2 = 0$ the ruled surfaces R_u of all curves through P (not tangent to the lines of curvature) have a common asymptotic tangent plane at P , namely, the osculating plane of C_u at P .

By direct substitution the coordinates of the center of Q_u , when finite, given by (38) can be shown to satisfy equation (34). This fact establishes the result:

The characteristic of an asymptotic tangent plane of a ruled surface R_u of a curve C passes through the center, when

finite of the quadric Q_u which is associated at P with the same curve C.

Later it is shown geometrically that the above theorem holds when the coordinates of the center of Q_u are not finite.

From equations (19), (34) and (35) the following theorems can be derived:

At points P where $1/\rho_s = 0$ all curves C on S through P tangent to a fixed direction determine ruled surfaces R_u whose common asymptotic tangent plane has a characteristic on which lie the centers of all quadrics Q_u defined at P for the curves C.

For the direction $\gamma = 0$ the centers of the quadrics Q_u and Q_v defined for this direction lie in the coordinate planes $\eta = 0$ and $\xi = 0$, respectively. If also $\alpha = 0$, the center of Q_u lies on the surface normal.

Eliminating λ' between equations (36) and (37), one obtains

$$(39) \quad \begin{aligned} &\psi\eta^2/G - (\phi + 2\alpha R_1/\rho_1)\zeta^2 - \lambda_{DT}\xi\eta \\ &- (\lambda/\rho_s - \lambda_{ET}/\rho_1)\xi\zeta - 2R_1\psi\eta\zeta/\rho_1G - \lambda^2\zeta = 0, \end{aligned}$$

where ψ , ϕ , T , α have the values given by (19). Since the quadric (39) intersects the tangent plane of the surface at the point P in two distinct tangents, the quadric (39) is a hyperbolic paraboloid or a hyperboloid of one sheet.

At a point P on a surface S the centers of all quadrics Q_u associated with a given direction lie in the plane (35) and on the quadric (39) defined for that direction. To determine the nature of the conic of intersection of (35) and (39) project the conic of intersection onto the η -plane. The equations of the conic are

$$(40) \quad \eta = 2\lambda\xi\zeta/\rho_2 - \zeta^2(\psi\gamma^2/GD^2T^2 - \phi - 2\alpha R_1/\rho_1 - 2R_1\psi\gamma/\rho_1 GDT) + \lambda\zeta = 0,$$

which may be written in the form

$$\eta = A\zeta = 0,$$

where the interpretation of A is obvious. Since the projection consists of two intersecting straight lines, the asymptotic tangent plane (35) is a tangent plane of the quadric (39). Therefore, the conic of intersection consists of two intersecting straight lines which are rulings of the quadric (39). From equations (35)-(37) it is evident that the center of Q_u cannot lie on the u -tangent. Therefore, the centers of the quadrics Q_u defined for all curves tangent to a given direction lie on the line

$$(41) \quad A = \gamma\zeta - DT\eta = 0,$$

which is a ruling of quadric (39). The fact that the line (41) always intersects the u -tangent at points where $1/\rho_2 \neq 0$ justifies the conclusion:

At points P where $1/\rho_2 \neq 0$ the locus of the ruling on which the centers of Q_u lie is a ruled surface.

To obtain some of the above results geometrically we polarize with respect to the quadric Q_u defined for the same curve C which defines R_u . The polar plane of the point at infinity on the generator of R_u which passes through the point P passes through the center of Q_u . Since the generator $\eta = \zeta = 0$ of R_u is also a generator of Q_u , the polar plane of the point at infinity on $\eta = \zeta = 0$ is the asymptotic tangent plane of the surface R_u . Since the u -tangent and a consecutive generator belong to the same regulus, they intersect at infinity. By the same argument

as above the asymptotic tangent plane of the point at infinity on the consecutive generator also passes through the center of the quadric Q_u . Consequently, the characteristic of the asymptotic tangent plane of R_u passes through the center of Q_u and is parallel to the u-tangent.

The equation of the line of intersection of the asymptotic tangent planes of R_u and R_v is

$$(42) \quad -\lambda \xi / R_2 \gamma = D^T \eta / \gamma = \zeta.$$

At points P where $1/\rho_1 \rho_2 \neq 0$ the locus of line (42) as λ varies is the cone

$$\eta \xi - R_1 \xi \zeta / \rho_1 + R_2 \eta \zeta / \rho_2 = 0,$$

which has the coordinate axes as generators.

If $(-a, -b, 1)$ are the coordinates of any point on a line passing through P, then the line generates a congruence whose developables have the differential equation¹

$$(43) \quad T_1 du^2 + (S_1 - P_1) du dv - Q_1 dv^2 = 0,$$

where

$$\begin{aligned} Q_1 &= G^{\frac{1}{2}} a_v + E^{\frac{1}{2}} b \{22, 1\} + D'' ab, \\ T_1 &= E^{\frac{1}{2}} b_u + G^{\frac{1}{2}} a \{11, 2\} + Dab, \end{aligned}$$

and where the values of S_1 and P_1 are omitted since they are not used.

A necessary and sufficient condition that the line (42) generate a congruence whose developables intersect the surface in a conjugate net is that λ satisfy the following equation:

¹E. P. Lane, Projective Differential Geometry of Curves and Surfaces, Exercise 10, p. 252.

$$D''T_1 - DQ_1 = 0,$$

where T_1 and Q_1 have the values given by $(43_{2,3})$ in which a and b are now to be replaced by

$$a = \gamma R_2 / \lambda, \quad b = -\gamma / DT.$$

The conjugate osculating quadrics on the lines of curvature at a point of a curve on a surface have the following property which is a special case of a more general result.¹ The conjugate osculating quadric Q_u at a point of a curve C on a non-developable surface S which is not a sphere and on which the lines of curvature are taken as parametric is the quadric of Lie at the point P on the ruled surface R_u formed by constructing the u -tangents of the lines of curvature at the points of C .

Similar results can be obtained for the quadric Q_v .

¹Let us consider a conjugate net N on a surface S in ordinary projective space, and on S any curve C not belonging to N . At the points of C let us construct the tangents of the u -curves of the net N . These tangents determine a ruled surface which we shall call R_u . The conjugate osculating quadric Q_u at a point P of the curve C on the net N on the surface S is the quadric of Lie at the point P of the ruled surface R_u . A similar result is true when the net N is replaced by the asymptotics and the conjugate osculating quadrics are replaced by the asymptotic osculating quadrics.

5. OTHER PROPERTIES

The equation of one quadric of a second pair of quadrics which are associated with the conjugate osculating quadrics at a point of a curve on the lines of curvature on a surface is derived in this section. The total and mean curvatures of a conjugate osculating quadric at a point of a curve on a surface are computed.

From the equation of the osculating plane at a point of a curve on a surface, namely, equation (14) the following theorem can be deduced immediately: If a curve on a surface is tangent to an asymptotic tangent at P, then either the osculating plane to C coincides with the tangent plane to the surface or C has an inflection at P. Therefore, when $\lambda \neq \pm(-D/D'')^{\frac{1}{2}}$, λ' can be eliminated between equations (14) and (18). The result is

$$(44) \quad 2D\eta^2/G + p\zeta^2 + q\xi\eta + r\xi\zeta + s\eta\zeta + 2\lambda^2\zeta = 0,$$

where

$$\begin{aligned} p &= 2E^2/\rho_1^2GD + 8\lambda\mathfrak{E}'R_1/\rho_1G^{\frac{1}{2}} + \lambda R_1R_2(3D + D''\lambda^2)T/\rho_1\rho_2 \\ &\quad + \lambda(-\lambda^2K/D + H/D'') + 2\lambda^2R_1/\rho_1^2 + 2\lambda^2R_1(1 - \rho_{su}/E^{\frac{1}{2}})/\rho_2^2, \\ q &= -\lambda T(3D + D''\lambda^2), \quad r = \lambda R_1(3D + D''\lambda^2)T/\rho_1 + 2\lambda^2/\rho_2, \\ s &= -4E/\rho_1G - 8\lambda\mathfrak{E}'/G^{\frac{1}{2}} - R_2\lambda T(3D + D''\lambda^2)/\rho_2, \quad T = 1/(EG)^{\frac{1}{2}}. \end{aligned}$$

We may formulate our results as follows:

For all directions, except those defined by $\lambda = 0, \infty$, $D + D''\lambda^2 = 0$, the osculating plane of a curve at a point of a surface intersects the conjugate osculating quadric Q_u associated with the same point of the curve in a conic. The locus of this conic as the curve varies but remains tangent to a fixed line is a quadric.

Since the discriminant Δ of quadric (44) is

$$\Delta = -\lambda^6 T^2 (3D + D'' \lambda^2),$$

the quadric (44) is singular for the directions given by

$$3D + D'' \lambda^2 = 0,$$

and for all other directions for which it is defined it is a hyperboloid of one sheet or a hyperbolic paraboloid.

The equation of the quadric Q_u at a point of a curve on a surface may be solved for ζ as a power series in terms of ξ, η . To this end write ζ as a power series in ξ, η with undetermined coefficients and then demand that this series when substituted for ζ in equation (18) of the quadric Q_u be satisfied identically in ξ, η as far as terms of the second degree. The solution is

$$(45) \quad \zeta = -[(D - D'' \lambda^2) \eta^{2/G} - 2\lambda D \xi \eta / E^{\frac{1}{2}G^{\frac{1}{2}}}] / 2\lambda^2 + \dots$$

When a surface is defined by $\zeta = f(\xi, \eta)$, the expressions for the curvatures¹ are

$$(46) \quad \begin{aligned} K_t &= (rt - s^2) / (1 + p^2 + q^2)^2, \\ K_m &= [(1 + p^2)t + (1 + q^2)r - 2pqs] / (1 + p^2 + q^2)^{3/2}, \end{aligned}$$

where K_t and K_m denote the total curvature and the mean curvature, respectively, and where

$$\begin{aligned} p &= \partial \zeta / \partial \xi, & q &= \partial \zeta / \partial \eta, & r &= \partial^2 \zeta / \partial \xi^2, \\ s &= \partial^2 \zeta / \partial \xi \partial \eta, & t &= \partial^2 \zeta / \partial \eta^2. \end{aligned}$$

Using formulas (46), (47), (45), and evaluating at the point P which has local coordinates (0, 0, 0), one obtains

¹L. P. Eisenhart, op. cit., Exercise 3, p. 126.

$$K_t = -D^2/\lambda^2 EG, \quad K_m = -(D - D''\lambda^2)/\lambda^2 G.$$

By equating the total curvatures of Q_u and the surface S at the point P and by equating the mean curvatures of Q_u and the surface S at the point P , the following results can be derived:

At a point of a surface which has negative total curvature the surface and the quadric Q_u defined for any curve tangent to an asymptotic at the point have the same total curvature.

At a point of a surface the surface and the quadric Q_u associated at the point with any curve tangent to the minimal directions, $E + G\lambda^2 = 0$, have the same mean curvature.

6. THE SUPEROSCULATING LINES

In this section an interpretation of the differential equation corresponding to the equation formed by setting equal to zero the terms of third degree in the series representing the surface at a point is given. It is also shown that a necessary and sufficient condition that the superosculating lines and the curves of Darboux coincide is that the surface has constant total curvature.

The power series expansions for the local coordinates ξ , η , ζ of a point near a point $P(0, 0, 0)$ on a surface S in terms of the increments Δu , Δv corresponding to displacement on S from the point $(0, 0, 0)$ to (ξ, η, ζ) are¹

$$\begin{aligned}
 \xi &= E^{\frac{1}{2}} [\Delta u + (\{11, 1\} \Delta u^2 + 2\{12, 1\} \Delta u \Delta v \\
 &\quad + \{22, 1\} \Delta v^2)/2 + \dots], \\
 \eta &= G^{\frac{1}{2}} [\Delta v + (\{11, 2\} \Delta u^2 + 2\{12, 2\} \Delta u \Delta v \\
 (47) \quad &\quad + \{22, 2\} \Delta v^2)/2 + \dots], \\
 \zeta &= (D \Delta u^2 + D'' \Delta v^2)/2 + [(D_u + \{11, 1\} D) \Delta u^3 \\
 &\quad + 3\{12, 1\} D \Delta u^2 \Delta v + 3\{12, 2\} D'' \Delta u \Delta v^2 \\
 &\quad + (D''_v + \{22, 2\} D'') \Delta v^3]/6 + \dots.
 \end{aligned}$$

To find an expansion for ζ as a power series in ξ , η set ζ equal to a power series in ξ , η with undetermined coefficients and then demand that the series (47) for ξ , η , ζ satisfy this series identically in Δu , Δv as far as the terms of any desired order. To terms of the third order the result is

¹Ibid., p. 239.

$$(48) \quad \zeta = (\xi^2/R_1 + \eta^2/R_2)/2 + [(1/R_1)_u \xi^3/E^{\frac{1}{2}} + 3(1/R_1)_v \xi^2 \eta/G^{\frac{1}{2}} + 3(1/R_2)_u \xi \eta^2/E^{\frac{1}{2}} + (1/R_2)_v \eta^3/G^{\frac{1}{2}}]/6 + \dots,$$

where $R_1 = E/D$ and $R_2 = G/D''$. The terms of third degree of series (48) set equal to zero yield

$$(49) \quad (1/R_1)_u \xi^3/E^{\frac{1}{2}} + 3(1/R_1)_v \xi^2 \eta/G^{\frac{1}{2}} + 3(1/R_2)_u \xi \eta^2/E^{\frac{1}{2}} + (1/R_2)_v \eta^3/G^{\frac{1}{2}} = 0.$$

The differential equation corresponding to equation (49) can be found by making the substitutions

$$(16) \quad \eta = \xi \tan \theta,$$

$$(17) \quad \tan \theta = v'(G/E)^{\frac{1}{2}} \quad (v' = dv/du)$$

in equation (48). After simplifying, the equation becomes

$$(50) \quad D^2(R_1)_u du^3/E + 3D^2(R_1)_v du^2 dv/E + 3D''^2(R_2)_u du dv^2/G + D''^2(R_2)_v dv^3/G = 0.$$

For the coordinate system described in Section 2 p_1 and q are defined¹ by

$$(51) \quad D = -qE^{\frac{1}{2}}, \quad D'' = p_1 G^{\frac{1}{2}}.$$

The result of substituting (51) in (50) is

$$(52) \quad q^2(R_1)_u du^3 + 3q^2(R_1)_v du^2 dv + 3p_1^2(R_2)_u du dv^2 + p_1^2(R_2)_v dv^3 = 0.$$

Conversely, equation (49) can be derived from equation (52). When a surface is referred to its lines of curvature, the curves defined by equation (52) are the superosculating lines on the sur-

¹Eisenhart, op. cit., p. 174.

face. They possess the property that the curves of normal section in these directions are straight lines, or are superosculated by their circles of curvature.¹ These results can be formulated as follows:

When a surface is represented by the series (48), equation (49) defines the directions of the superosculating lines on the surface.

The differential equation of the curves of Darboux² on a surface S is

$$(52) \quad D\mathcal{G}'du^3 - 3D\mathcal{B}'du^2dv - 3D''\mathcal{G}'du dv^2 + D''\mathcal{B}'dv^3 = 0.$$

The superosculating lines are the curves of Darboux, if and only if

$$(53) \quad \begin{aligned} m\mathcal{G}' &= (\log R_1)_u, & -m\mathcal{B}' &= (\log R_1)_v, \\ -m\mathcal{G}' &= (\log R_2)_u, & m\mathcal{B}' &= (\log R_2)_v, \end{aligned}$$

where m is a proportionality factor. From equations (53) it follows that the surface has constant total curvature. Conversely, if a surface has constant total curvature, then the formulas (12) for the metric invariants become

$$\begin{aligned} 2\mathcal{B}' &= (\log R_1)_v = -(\log R_2)_v, \\ 2\mathcal{G}' &= (\log R_2)_u = -(\log R_1)_u. \end{aligned}$$

Therefore, the superosculating lines on a surface are the curves of Darboux on the surface if and only if the surface has constant total curvature.

¹Ibid., Exercise 13, p. 187.

²E. P. Lane, Projective Differential Geometry of Curves and Surfaces, p. 240.

VITA

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